

EQUATIONS INVOLVING FRACTIONAL LAPLACIAN OPERATOR: COMPACTNESS AND APPLICATION

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Abstract. In this paper, we consider the following problem involving fractional Laplacian operator:

$$(-\Delta)^\alpha u = |u|^{2_\alpha^*-2-\varepsilon} u + \lambda u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (1)$$

where Ω is a smooth bounded domain in \mathbb{R}^N , $\varepsilon \in [0, 2_\alpha^* - 2)$, $0 < \alpha < 1$, $2_\alpha^* = \frac{2N}{N-2\alpha}$. We show that for any sequence of solutions u_n of (1) corresponding to $\varepsilon_n \in [0, 2_\alpha^* - 2)$, satisfying $\|u_n\|_H \leq C$ in the Sobolev space H defined in (1.2), u_n converges strongly in H provided that $N > 6\alpha$ and $\lambda > 0$. An application of this compactness result is that problem (1) possesses infinitely many solutions under the same assumptions.

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1. INTRODUCTION

In this paper, we consider the following problem with the fractional Laplacian:

$$\begin{cases} (-\Delta)^\alpha u = |u|^{2_\alpha^*-2-\varepsilon} u + \lambda u & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a smooth bounded domain in \mathbb{R}^N , $\varepsilon \in [0, 2_\alpha^* - 2)$, $\lambda > 0$, $0 < \alpha < 1$, and $2_\alpha^* = \frac{2N}{N-2\alpha}$ is the critical exponent in fractional Sobolev inequalities.

In a bounded domain $\Omega \subset \mathbb{R}^N$, we define the operator $(-\Delta)^\alpha$ as follows. Let $\{\lambda_k, \varphi_k\}_{k=1}^\infty$ be the eigenvalues and corresponding eigenfunctions of the Laplacian operator $-\Delta$ in Ω with zero Dirichlet boundary values on $\partial\Omega$ normalized by $\|\varphi_k\|_{L^2(\Omega)} = 1$, i.e.

$$-\Delta \varphi_k = \lambda_k \varphi_k \quad \text{in } \Omega; \quad \varphi_k = 0 \quad \text{on } \partial\Omega.$$

For any $u \in L^2(\Omega)$, we may write

$$u = \sum_{k=1}^{\infty} u_k \varphi_k, \quad \text{where} \quad u_k = \int_{\Omega} u \varphi_k dx.$$

We define the space

$$H = \left\{ u = \sum_{k=1}^{\infty} u_k \varphi_k \in L^2(\Omega) : \sum_{k=1}^{\infty} \lambda_k^\alpha u_k^2 < \infty \right\}, \quad (1.2)$$

which is equipped with the norm

$$\|u\|_H = \left(\sum_{k=1}^{\infty} \lambda_k^\alpha u_k^2 \right)^{\frac{1}{2}}.$$

For any $u \in H$, the fractional Laplacian $(-\Delta)^\alpha$ is defined by

$$(-\Delta)^\alpha u = \sum_{k=1}^{\infty} \lambda_k^\alpha u_k \varphi_k.$$

With this definition, we see that problem (1.1) is the Brézis-Nirenberg type problem with the fractional Laplacian. In [5], Brézis and Nirenberg considered the existence of positive solutions for problem (1.1) with $\alpha = 1$ and $\varepsilon = 0$. Such a problem involves the critical Sobolev exponent $2^* = \frac{2N}{N-2}$ for $N \geq 3$, and it is well known that the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact even if Ω is bounded. Hence, the associated functional of problem (1.1) does not satisfy the Palais-Smale

condition, and critical point theory cannot be applied directly to find solutions of the problem. However, it is found in [5] that the functional satisfies the $(PS)_c$ condition for $c \in (0, \frac{1}{N}S^{\frac{N}{2}})$, where S is the best Sobolev constant and $\frac{1}{N}S^{\frac{N}{2}}$ is the least level at which the Palais-Smale condition fails. So a positive solution can be found if the mountain pass value corresponding to problem (1.1) is strictly less than $\frac{1}{N}S^{\frac{N}{2}}$. In [18], a concentration-compactness principle was developed to treat non-compact critical variational problems. In the study of the existence of multiple solutions for critical problems, to retain the compactness, it is necessary to have a full description of energy levels at which the associated functional does not satisfy the Palais-Smale condition. A global compactness result is found in [21], which describes precisely the obstacles of the compactness for critical semilinear elliptic problems. This compactness result shows that above certain energy level, it is impossible to prove the Palais-Smale condition. For this reason, to obtain many solutions for the critical problem, it is essential to find a condition that can replace the standard Palais-Smale condition.

In [14], Devillanova and Solimini considered (1.1) with $\alpha = 1$. They started by considering any sequence of solutions u_n of (1.1) corresponding to $\varepsilon_n > 0$, $\varepsilon_n \rightarrow 0$, satisfying $\|u_n\|_H \leq C$ in the Sobolev space H defined in (1.2). By analyzing the bubbling behaviors of u_n , they are able to show that u_n converges strongly to a solution of the critical problem in H if $N > 7$ and $\lambda > 0$. A consequence of this compactness result is that (1.1) with $\alpha = 1$ is that (1.1) with $\alpha = 1$ and $\varepsilon = 0$ has infinitely many solutions. So, we see that the compactness of the solutions set for (1.1) can be used to replace the Palais-Smale condition in the critical point theories.

Let us point out that the same idea was used in [12], [13] and [26] to study other problems involving critical exponents, though the methods used in [12, 13, 26] to obtain the estimates are different from those in [14].

Problems with the fractional Laplacian have been extensively studied recently. See for example [3, 4, 6, 7, 8, 10, 11, 16, 20, 22, 23, 24]. In particular, the Brézis-Nirenberg type problem was discussed in [23] for the special case $\alpha = \frac{1}{2}$, and in [4] for the general case, $0 < \alpha < 1$, where existence of one positive solution was proved. To use the idea in [5] to prove the existence of one positive solution for the fractional Laplacian, the authors in [4, 23] used the following results in [11] (see also [3]): for any $u \in H$, the solution $v \in H_{0,L}^1(\mathcal{C}_\Omega)$ of the problem

$$\begin{cases} -\operatorname{div}(y^{1-2\alpha}\nabla v) = 0, & \text{in } \mathcal{C}_\Omega = \Omega \times (0, \infty), \\ v = 0, & \text{on } \partial_L \mathcal{C}_\Omega = \partial\Omega \times (0, \infty), \\ v = u, & \text{on } \Omega \times \{0\}, \end{cases} \quad (1.3)$$

satisfies

$$-\lim_{y \rightarrow 0^+} k_\alpha y^{1-2\alpha} \frac{\partial v}{\partial y} = (-\Delta)^\alpha u,$$

where we use $(x, y) = (x_1, \dots, x_N, y) \in \mathbb{R}^{N+1}$, and

$$H_{0,L}^1(\mathcal{C}_\Omega) = \{v \in L^2(\mathcal{C}_\Omega) : v = 0 \text{ on } \partial_L \mathcal{C}_\Omega, \int_{\mathcal{C}_\Omega} y^{1-2\alpha} |\nabla v|^2 dx dy < \infty\}. \quad (1.4)$$

Therefore, the nonlocal problem (1.1) can be reformulated to the following local problem:

$$\begin{cases} -\operatorname{div}(y^{1-2\alpha} \nabla v) = 0, & \text{in } \mathcal{C}_\Omega, \\ v = 0, & \text{on } \partial_L \mathcal{C}_\Omega, \\ y^{1-2\alpha} \frac{\partial v}{\partial \nu} = |v(x, 0)|^{2_\alpha^* - 2 - \varepsilon} v(x, 0) + \lambda v(x, 0), & \text{on } \Omega \times \{0\}, \end{cases} \quad (1.5)$$

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative of $\partial \mathcal{C}_\Omega$. Hence, critical points of the functional

$$I_\varepsilon(v) = \frac{1}{2} \int_{\mathcal{C}_\Omega} y^{1-2\alpha} |\nabla v|^2 dx dy - \frac{1}{2_\alpha^* - \varepsilon} \int_{\Omega \times \{0\}} |v|^{2_\alpha^* - \varepsilon} dx - \frac{\lambda}{2} \int_{\Omega \times \{0\}} |v|^2 dx \quad (1.6)$$

defined on $H_{0,L}^1(\mathcal{C}_\Omega)$ correspond to solutions of (1.5). A solution at the mountain pass level of the functional $I(u)$ was found in [4, 23]. On the other hand, it is easy to show by using the Pohozaev type identity that the problem

$$(-\Delta)^\alpha u = |u|^{p-1} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

has no nontrivial solution if $p + 1 \geq \frac{2N}{N-2\alpha}$ and Ω is star-shaped.

In this paper, we will investigate the existence of infinitely many solutions for problem (1.1) by finding critical points of the functional $I(u)$. Since the problem is critical, the functional $I(u)$ does not satisfy the Palais-Smale condition. Thus the mini-max theorems can not be applied directly to obtain infinitely many solutions for (1.1). So we follow the idea in [14] to consider the subcritical problem

$$\begin{cases} \operatorname{div}(y^{1-2\alpha} \nabla v) = 0, & \text{in } \mathcal{C}_\Omega, \\ v = 0, & \text{on } \partial_L \mathcal{C}_\Omega, \\ y^{1-2\alpha} \frac{\partial v}{\partial y} = -|v(x, 0)|^{p_n - 2} v(x, 0) - \lambda v(x, 0), & \text{on } \Omega \times \{0\}. \end{cases} \quad (1.7)$$

where $p_n = 2_\alpha^* - \varepsilon_n$ with $\varepsilon_n \rightarrow 0$.

The main result of this paper is the following.

Theorem 1.1. *Suppose $N > 6\alpha$, then for any v_n , which is a solution of (1.7) satisfying $\|v_n\|_{H_{0,L}^1(\mathcal{C}_\Omega)} \leq C$ for some constant independent of n , v_n converges strongly in $H_{0,L}^1(\mathcal{C}_\Omega)$ as $n \rightarrow +\infty$.*

Theorem 1.1 is a special compactness result. It shows that although $I(u)$ does not satisfy the Palais–Smale condition, for a special Palais–Smale sequence, which is solutions of the perturbed problem (1.7), it does converge strongly in $H_{0,L}^1(\mathcal{C}_\Omega)$. It is well known now [9, 12] that this weak compactness leads to the following existence result:

Theorem 1.2. *If $N > 6\alpha$, then (1.1) with $\varepsilon = 0$ has infinitely many solutions.*

The main difficulty in the study of (1.7) is that we need to carry out the boundary estimates. This is different from the Dirichlet problems studied in [9, 12, 13, 14, 26], which mainly involve the interior estimates.

This paper is organized as follows. In section 2, we will state a decomposition result for the solutions of the perturbed problem (1.7). In section 3, we obtain some integral estimates which captures the possible bubbling behavior of the solutions of (1.7). To prove such estimates, we need to study a linear problem. This part is of independent interest. So we put it in Appendix A. Section 4 contains the estimates for solutions of (1.7) in the region which does not contain any blow up point, but is close to some blow up point. The main result is proved in section 5 by using the local Pohozaev identity, together with the estimates in section 4. In Appendix B, we prove a decay estimate for solutions of a problem in half space involving the fractional critical Sobolev exponent.

Throughout this paper, we use $\mathcal{B}_r(z)$ to denote the ball in \mathbb{R}^{N+1} , centered at $z \in \mathbb{R}^{N+1}$ with radius r . We also use $X = (x, y)$ to denote a point in \mathbb{R}^{N+1} , and for any set $D \in \mathbb{R}^N$,

$$\mathcal{C}_D = D \times (0, \infty) \subset \mathbb{R}^{N+1}, \quad \partial_L \mathcal{C}_D = \partial D \times (0, +\infty). \quad (1.8)$$

2. PRELIMINARIES

Let Ω be a smooth bounded domain in \mathbb{R}^N and $0 < \alpha < 1$. The space $H^\alpha(\Omega)$ is defined as the subset of $L^2(\Omega)$ such that for $u \in L^2(\Omega)$, the norm

$$\|u\|_{H^\alpha(\Omega)} = \|u\|_{L^2(\Omega)} + \left(\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(\tilde{x})|^2}{|x - \tilde{x}|^{N+2\alpha}} dx d\tilde{x} \right)^{\frac{1}{2}}$$

is finite. Let $H_0^\alpha(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{H^\alpha(\Omega)}$. It is known from [17] that for $0 < \alpha \leq \frac{1}{2}$, $H_0^\alpha(\Omega) = H^\alpha(\Omega)$; for $\frac{1}{2} < \alpha < 1$, $H_0^\alpha(\Omega) \subsetneq H^\alpha(\Omega)$.

The space H defined in (1.2) is the interpolation space $(H_0^2(\Omega), L^2(\Omega))_{\alpha,2}$, see [1, 17, 25]. It was shown in [17] that $(H_0^2(\Omega), L^2(\Omega))_{\alpha,2} = H_0^\alpha(\Omega)$ if $0 < \alpha < 1$ and $\alpha \neq \frac{1}{2}$; while $(H_0^2(\Omega), L^2(\Omega))_{\frac{1}{2},2} = H_{00}^{\frac{1}{2}}(\Omega)$, where

$$H_{00}^{\frac{1}{2}}(\Omega) = \{u \in H^{\frac{1}{2}}(\Omega) : \int_{\Omega} \frac{u^2(x)}{d(x)} dx < \infty\},$$

and $d(x) = \text{dist}(x, \partial\Omega)$ for all $x \in \Omega$. We know from [4], see also [8], that for any $u \in H_0^\alpha(\Omega)$, let $v \in H_{0,L}^1(\mathcal{C}_\Omega)$ be the extension of u defined in (1.3), then the mapping $u \rightarrow v$ is an isometry between $H_0^\alpha(\Omega)$ and $H_{0,L}^1(\mathcal{C}_\Omega)$. That is

$$\|v\|_{H_{0,L}^1(\mathcal{C}_\Omega)} = \|u\|_{H_0^\alpha(\Omega)} \quad \text{for } u \in H_0^\alpha(\Omega).$$

For any function W defined on \mathbb{R}^{N+1} , $x \in \mathbb{R}^N$, $\sigma > 0$, we define

$$\rho_{x,\sigma}(W) = \sigma^{\frac{N-2\alpha}{2}} W(\sigma(\cdot - (x, 0))). \quad (2.1)$$

It is now standard to prove the following decomposition result.

Proposition 2.1. *Let $\{v_n\} \subset H_{0,L}^1(\mathcal{C}_\Omega)$ be a sequence of solutions of (1.7) satisfying $\|v_n\|_{H_{0,L}^1(\mathcal{C}_\Omega)} \leq C$. Then, there exist a solution $v_0 \in H_{0,L}^1(\mathcal{C}_\Omega)$ of (1.5), a finite sequence $\{W^j\}_{j=1}^k \subset H_{0,L}^1(\mathbb{R}^N)$, which are solutions of*

$$\begin{cases} \text{div}(y^{1-2\alpha} \nabla v) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\ y^{1-2\alpha} \frac{\partial v}{\partial y} = -\beta_j |v(x, 0)|^{2_\alpha^* - 2} v(x, 0), & \text{in } \mathbb{R}^N, \end{cases} \quad (2.2)$$

where $\beta_j \in (0, 1]$ is some constant, and sequences $\{x_n^j\}_{j=1}^k$, $\{\sigma_n^j\}_{j=1}^k$ satisfying $\sigma_n^j > 0$, $x_n^j \in \Omega$ and as $n \rightarrow +\infty$,

$$\sigma_n^j \text{dist}(x_n^j, \partial\Omega) \rightarrow \infty, \quad \frac{\sigma_n^j}{\sigma_n^i} + \frac{\sigma_n^i}{\sigma_n^j} + \sigma_n^i \sigma_n^j |x_n^i - x_n^j|^2 \rightarrow +\infty, \quad i \neq j, \quad (2.3)$$

$$\|v_n - v_0 - \sum_{j=1}^k \rho_{x_n^j, \sigma_n^j}(W^j)\|_{H_{0,L}^1(\mathbb{R}^N)} \rightarrow 0. \quad (2.4)$$

3. INTEGRAL ESTIMATES

To prove Theorem 1.1, we need to prove that the bubbles $\rho_{x_n^j, \sigma_n^j}(W^j)$ do not appear in the decomposition (2.4).

Similar to [14], we introduce the following norm. Let $q_1, q_2 \in (2, \infty)$ be such that $q_2 < 2_\alpha^* < q_1$, $\beta > 0$ and $\sigma > 0$. We consider the following inequalities

$$\begin{cases} \|u_1\|_{q_1} \leq \beta, \\ \|u_2\|_{q_2} \leq \beta \sigma^{\frac{N}{2_\alpha^*} - \frac{N}{q_2}} \end{cases} \quad (3.1)$$

and define the norm

$$\|u\|_{q_1, q_2, \sigma} = \inf \{ \beta > 0 : \text{there exist } u_1, u_2 \text{ such that (3.1) holds and } |u| \leq u_1 + u_2 \}. \quad (3.2)$$

Denote

$$\sigma_n = \min_{1 \leq j \leq k} \sigma_n^j.$$

In this section, we will prove the following result.

Proposition 3.1. *Let v_n be a solution of (1.7). For any $q_1, q_2 \in (\frac{N}{N-2\alpha}, +\infty)$, $q_2 < 2_\alpha^* < q_1$, there is a constant $C > 0$, depending only on q_1 and q_2 , such that*

$$\|v_n\|_{q_1, q_2, \sigma_n} \leq C. \quad (3.3)$$

To prove Proposition 3.1, it is convenient to consider the following problem. Let D be a bounded domain such that $\Omega \subset\subset D$ and let $v_n(x, 0) = 0$ in $D \setminus \Omega$. We choose $A > 0$ large enough so that

$$| |t|^{p_n-2}t + \lambda t | \leq 2|t|^{2_\alpha^*-1} + A, \quad \forall t \in \mathbb{R}.$$

Solving

$$\begin{cases} \operatorname{div}(y^{1-2\alpha} \nabla w) = 0, & \text{in } \mathcal{C}_D, \\ w = 0, & \text{on } \partial_L \mathcal{C}_D, \\ y^{1-2\alpha} \frac{\partial w}{\partial \nu} = 2|v_n(x, 0)|^{2_\alpha^*-1} + A, & \text{on } D \times \{0\}, \end{cases} \quad (3.4)$$

we obtain a sequence of solutions $\{w_n\}$ with $w_n \geq 0$. By the choice of D and A , we find

$$\begin{cases} \operatorname{div}(y^{1-2\alpha}\nabla(w_n \pm v_n)) = 0, & \text{in } \mathcal{C}_\Omega, \\ w_n \pm v_n \geq 0, & \text{on } \partial_L \mathcal{C}_\Omega, \\ y^{1-2\alpha} \frac{\partial(w_n \pm v_n)}{\partial \nu} \geq 0, & \text{on } \Omega \times \{0\}. \end{cases} \quad (3.5)$$

Multiplying (3.5) by $(w_n \pm v_n)^-$ and integrating by part, we see that

$$|v_n| \leq w_n, \quad \text{in } \mathcal{C}_\Omega.$$

Hence, it is sufficient to estimate w_n in \mathcal{C}_D .

Lemma 3.1. *Let $w \in H_{0,L}^1(\mathcal{C}_D)$ be a solution of*

$$\begin{cases} \operatorname{div}(y^{1-2\alpha}\nabla w) = 0 & \text{in } \mathcal{C}_D, \\ w = 0 & \text{on } \partial_L \mathcal{C}_D, \\ y^{1-2\alpha} \frac{\partial w}{\partial \nu} = a(x)v & \text{on } D \times \{0\}, \end{cases} \quad (3.6)$$

where $a \in L^{\frac{N}{2\alpha}}(D)$, $v \in C^\beta(D)$ and $a, v \geq 0$. For any $q_1, q_2 \in (\frac{N}{N-2\alpha}, +\infty)$, $q_2 < 2_\alpha^* < q_1$, there exists $C = C(N, q_1, q_2) > 0$, such that

$$\|w(\cdot, 0)\|_{q_1, q_2, \sigma} \leq C \|a\|_{L^{\frac{N}{2\alpha}}(D)} \|v\|_{q_1, q_2, \sigma}. \quad (3.7)$$

Proof. For any $\varepsilon > 0$ small and $\sigma > 0$ fixed, let $v_1 \geq 0$ and $v_2 \geq 0$ be functions such that $|v| \leq v_1 + v_2$ and satisfying (3.1) with $\beta = \|v\|_{q_1, q_2, \sigma} + \varepsilon$. For $i = 1, 2$, consider

$$\begin{cases} \operatorname{div}(y^{1-2\alpha}\nabla w_i) = 0 & \text{in } \mathcal{C}_D, \\ w_i = 0 & \text{on } \partial_L \mathcal{C}_D, \\ y^{1-2\alpha} \frac{\partial w_i}{\partial \nu} = a(x)v_i & \text{on } D \times \{0\}. \end{cases} \quad (3.8)$$

By Corollary A.1,

$$\|w_i(\cdot, 0)\|_{L^{q_i}(D)} \leq C \|a\|_{L^{\frac{N}{2\alpha}}(D)} \|v_i\|_{L^{q_i}(D)}, \quad i = 1, 2. \quad (3.9)$$

On the other hand, it follows from the comparison theorem that

$$0 \leq w \leq w_1 + w_2,$$

since $|v| \leq v_1 + v_2$. Thus we complete the proof. \square

Lemma 3.2. *Let $w > 0$ be the solution of*

$$\begin{cases} \operatorname{div}(y^{1-2\alpha}\nabla w) = 0, & \text{in } \mathcal{C}_D, \\ w = 0, & \text{on } \partial_L \mathcal{C}_D, \\ y^{1-2\alpha}\frac{\partial w}{\partial \nu} = 2|v(x, 0)|^{2_\alpha^*-1} + A, & \text{on } D \times \{0\}, \end{cases} \quad (3.10)$$

where $v \in C^\beta(D)$ is a nonnegative function. Suppose $p_1, p_2 \in (\frac{N+2\alpha}{N-2\alpha}, \frac{N}{2\alpha}\frac{N+2\alpha}{N-2\alpha})$ and $p_2 < 2_\alpha^* < p_1$. Let q_1, q_2 be determined by

$$\frac{1}{q_i} = \frac{N+2\alpha}{N-2\alpha}\frac{1}{p_i} - \frac{2\alpha}{N}, \quad i = 1, 2. \quad (3.11)$$

Then, there exists a constant $C = C(N, p_1, p_2, \Omega) > 0$ such that for any $\sigma > 0$, it holds

$$\|w(\cdot, 0)\|_{q_1, q_2, \sigma} \leq C(\|v\|_{p_1, p_2, \sigma}^{\frac{N+2\alpha}{N-2\alpha}} + 1).$$

Proof. Choose $v_1 \geq 0$ and $v_2 \geq 0$, with $|v| \leq v_1 + v_2$ and

$$\|v_1\|_{L^{p_1}(D)} \leq (\|v\|_{p_1, p_2, \sigma} + \varepsilon), \quad \|v_2\|_{L^{p_2}(D)} \leq \sigma^{\frac{N}{2_\alpha^*} - \frac{N}{p_2}} (\|v\|_{p_1, p_2, \sigma} + \varepsilon).$$

Now we consider the following problems

$$\begin{cases} -\operatorname{div}(y^{1-2\alpha}\nabla w_1) = 0 & \text{in } \mathcal{C}_D, \\ w_1 = 0 & \text{on } \partial_L \mathcal{C}_D, \\ y^{1-2\alpha}\frac{\partial w_1}{\partial \nu} = 2^{\frac{4\alpha}{N-2\alpha}} v_1^{\frac{N+2\alpha}{N-2\alpha}} + A & \text{on } D \times \{0\}, \end{cases} \quad (3.12)$$

and

$$\begin{cases} -\operatorname{div}(y^{1-2\alpha}\nabla w_2) = 0 & \text{in } \mathcal{C}_D, \\ w_2 = 0 & \text{on } \partial_L \mathcal{C}_D, \\ y^{1-2\alpha}\frac{\partial w_2}{\partial \nu} = 2^{\frac{4\alpha}{N-2\alpha}} v_2^{\frac{N+2\alpha}{N-2\alpha}} & \text{on } D \times \{0\}. \end{cases} \quad (3.13)$$

Since

$$|v|^{\frac{N+2\alpha}{N-2\alpha}} \leq 2^{\frac{4\alpha}{N-2\alpha}} v_1^{\frac{N+2\alpha}{N-2\alpha}} + 2^{\frac{4\alpha}{N-2\alpha}} v_2^{\frac{N+2\alpha}{N-2\alpha}},$$

by comparison, $0 \leq w \leq w_1 + w_2$. Hence, we need to estimate $\|w_1(\cdot, 0)\|_{L^{q_1}(D)}$ and $\|w_2(\cdot, 0)\|_{L^{q_2}(D)}$. Since $1 < p_i \frac{N-2\alpha}{N+2\alpha} < \frac{N}{2\alpha}$, by Proposition A.1,

$$\begin{aligned} & \|w_1(\cdot, 0)\|_{L^{q_1}(D)} \\ & \leq C(N, p_1) \|v_1\|_{L^{\frac{N+2\alpha}{N-2\alpha}}(D)}^{\frac{N+2\alpha}{N-2\alpha}} + A \|v_1\|_{L^{p_1 \frac{N-2\alpha}{N+2\alpha}}(D)} \\ & \leq C(N, p_1) \left(\|v_1\|_{L^{p_1}(D)}^{\frac{N+2\alpha}{N-2\alpha}} + A |D|^{\frac{1}{p_1} \frac{N+2\alpha}{N-2\alpha}} \right) \\ & \leq C(N, p_1, D) \left((\|v\|_{p_1, p_2, \sigma} + \varepsilon)^{\frac{N+2\alpha}{N-2\alpha}} + 1 \right). \end{aligned}$$

Similarly, we have

$$\|w_2(\cdot, 0)\|_{L^{q_2}(D)} \leq C \|v_2\|_{L^{p_2}(D)}^{\frac{N+2\alpha}{N-2\alpha}} \leq C (\|v\|_{p_1, p_2, \sigma} + \varepsilon)^{\frac{N+2\alpha}{N-2\alpha}} \sigma^{\left(\frac{N}{2\alpha^*} - \frac{N}{p_2}\right) \frac{N+2\alpha}{N-2\alpha}}.$$

Since

$$\left(\frac{N}{2\alpha^*} - \frac{N}{p_2}\right) \frac{N+2\alpha}{N-2\alpha} = \frac{N}{2\alpha^*} - \frac{N}{q_2},$$

w_1, w_2 satisfies (3.1) with $\alpha = C \left((\|v\|_{p_1, p_2, \sigma} + \varepsilon)^{\frac{N+2\alpha}{N-2\alpha}} + 1 \right)$. The proof is completed by letting $\varepsilon \rightarrow 0$. □

Lemma 3.3. *Let w_n be a solution of (3.4). There are constants $C > 0$, $q_1, q_2 \in (\frac{N}{N-2\alpha}, +\infty)$, $q_2 < 2\alpha^* < q_1$, such that*

$$\|w_n\|_{q_1, q_2, \sigma_n} \leq C. \quad (3.14)$$

Proof. Since $\{\|v_n\|_{H_{0,L}^1(\mathcal{C}_\Omega)}\}$ is uniformly bounded, we may assume $v_n \rightharpoonup v_0$. By Proposition 2.1, we may write $v_n = v_0 + v_{n,1} + v_{n,2}$, where

$$v_{n,1}(x, y) = \sum_{j=1}^k \rho_{x_n^j, \sigma_n^j}(W_j)$$

$v_{n,2} = v_n - v_0 - v_{n,1}$. Let $a_0 = C|v_0|^{\frac{4\alpha}{N-2\alpha}}$ and $a_i = C|v_{n,i}|^{\frac{4\alpha}{N-2\alpha}}$, $i = 1, 2$ for $C > 0$ large.

Denote by $w = G(v)$ the solution of the following problem

$$\begin{cases} -\operatorname{div}(y^{1-2\alpha}\nabla w) = 0 & \text{in } \mathcal{C}_D, \\ w = 0 & \text{on } \partial_L \mathcal{C}_D, \\ y^{1-2\alpha}\frac{\partial w}{\partial \nu} = v & \text{on } D \times \{0\}. \end{cases} \quad (3.15)$$

By the comparison theorem,

$$w_n \leq G(a_0(\cdot, 0)|v_n(\cdot, 0)| + A) + G(a_1(\cdot, 0)|v_n(\cdot, 0)|) + G(a_2(\cdot, 0)|v_n(\cdot, 0)|).$$

Note that $v_0 \in L^\infty(\Omega)$. So $a_0 \in L^\infty(D)$. Taking $\frac{2N}{N+2\alpha} < p < 2_\alpha^*$, since $N > 2\alpha$, we have $\frac{N}{2\alpha} > 2_\alpha^*$, and then $q_1 := \frac{Np}{N-2\alpha p} > 2_\alpha^*$. By Proposition A.1 and Hölder's inequality,

$$\begin{aligned} & \|G(a_0(\cdot, 0)|v_n(\cdot, 0)| + A)(\cdot, 0)\|_{L^{q_1}(D)} \\ & \leq C\|v_n(\cdot, 0)\|_{L^p(D)} + C \leq C\|v_n(\cdot, 0)\|_{L^{2_\alpha^*}(D)} + C \leq C. \end{aligned}$$

This implies that for any $q_2 < 2_\alpha^*$,

$$\|G(a_0(\cdot, 0)|v_n(\cdot, 0)| + A)(\cdot, 0)\|_{q_1, q_2, \sigma_n} \leq \|G(a_0(\cdot, 0)|v_n(\cdot, 0)| + A)(\cdot, 0)\|_{L^{q_1}(D)} \leq C.$$

To estimate $G(a_1(\cdot, 0)|v_n(\cdot, 0)|)(\cdot, 0)$, we choose r such that $\frac{N}{4\alpha} < r < \frac{N}{2\alpha}$ and $\frac{1}{q_2} = \frac{1}{r} + \frac{1}{2_\alpha^*} - \frac{2\alpha}{N}$, we have $\frac{2N}{N+2\alpha} < q_2 < 2_\alpha^*$. By Corollary A.2,

$$\|G(a_1(\cdot, 0)|v_n(\cdot, 0)|)(\cdot, 0)\|_{L^{q_2}(D)} \leq C\|a_1(\cdot, 0)\|_{L^r(\Omega)}\|v_n(\cdot, 0)\|_{L^{2_\alpha^*}(\Omega)}.$$

Noting that $\frac{N-2\alpha r}{r} = (\frac{1}{q_2} - \frac{1}{2_\alpha^*})N$, we find

$$\|a_1(\cdot, 0)\|_{L^r(\Omega)} \leq \sum_{j=1}^k (\sigma_n^j)^{-\frac{N-2\alpha r}{r}} \left(\int_{\mathbb{R}^N} |W^j|^{\frac{4r\alpha}{N-2\alpha}} dx \right)^{\frac{1}{r}} \leq C\sigma_n^{\frac{N}{2_\alpha^*} - \frac{N}{q_2}},$$

since, by Proposition B.1,

$$|W^j|^{\frac{4r\alpha}{N-2\alpha}} \leq \frac{C}{(1 + |X|)^{4r\alpha}}$$

and $4r\alpha > N$. Therefore,

$$\|G(a_1(\cdot, 0)w_n(\cdot, 0))(\cdot, 0)\|_{q_1, q_2, \sigma_n} \leq \|G(a_1(\cdot, 0)w_n(\cdot, 0))(\cdot, 0)\|_{L^{q_2}(\Omega)}\sigma_n^{\frac{N}{q_2} - \frac{N}{2_\alpha^*}} \leq C.$$

Using Lemma 3.1, we deduce

$$\begin{aligned} & \|G(a_2(\cdot, 0)|v_n(\cdot, 0))(\cdot, 0)\|_{q_1, q_2, \sigma_n} \\ & \leq \|a_2(\cdot, 0)\|_{L^{\frac{N}{2\alpha}}(\Omega)} \|v_n(\cdot, 0)\|_{q_1, q_2, \sigma_n} \leq \frac{1}{2} \|w_n(\cdot, 0)\|_{q_1, q_2, \sigma_n}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \|w_n(\cdot, 0)\|_{q_1, q_2, \sigma_n} \\ & \leq 2\|G(a_0(\cdot, 0)w_n(\cdot, 0))(\cdot, 0)\|_{q_1, q_2, \sigma_n} + 2\|G(a_1(\cdot, 0)w_n(\cdot, 0))(\cdot, 0)\|_{q_1, q_2, \sigma_n} \\ & \leq C. \end{aligned}$$

The proof is complete. \square

Proof of Proposition 3.1. Since $|v_n| \leq w_n$, by Lemmas 3.2 and 3.3, for any constants $q_1, q_2 \in (\frac{N}{N-2\alpha}, +\infty)$, $q_2 < 2_\alpha^* < q_1$, it holds

$$\|v_n\|_{q_1, q_2, \sigma_n} \leq \|w_n\|_{q_1, q_2, \sigma_n} \leq C. \quad (3.16)$$

So the result follows. \square

4. ESTIMATES ON SAFE REGIONS

Since $\|v_n\|_E$ is uniformly bounded in n , the number of the bubble of v_n is also uniformly bounded in n , and we can find a constant $\bar{C} > 0$, independent of n , such that the region

$$\mathcal{A}_n^1 = \{X = (x, y) : X \in \left(\mathcal{B}_{(\bar{C}+5)\sigma_n^{-\frac{1}{2}}}(x_n, 0) \setminus \mathcal{B}_{\bar{C}\sigma_n^{-\frac{1}{2}}}(x_n, 0) \right) \cap \mathcal{C}_\Omega\}$$

does not contain any concentration point of v_n for any n , where $\mathcal{B}_r(z)$ is the ball in \mathbb{R}^{N+1} centered at z with the radius r . We call \mathcal{A}_n^1 safe region. Let

$$\mathcal{A}_n^2 = \{X : X \in \left(\mathcal{B}_{(\bar{C}+4)\sigma_n^{-\frac{1}{2}}}(x_n, 0) \setminus \mathcal{B}_{(\bar{C}+1)\sigma_n^{-\frac{1}{2}}}(x_n, 0) \right) \cap \mathcal{C}_\Omega\}$$

and

$$\mathcal{A}_n^3 = \{X : X \in \left(\mathcal{B}_{(\bar{C}+3)\sigma_n^{-\frac{1}{2}}}(x_n, 0) \setminus \mathcal{B}_{(\bar{C}+2)\sigma_n^{-\frac{1}{2}}}(x_n, 0) \right) \cap \mathcal{C}_\Omega\}.$$

In this section, we will prove the following result.

Proposition 4.1. *There is a constant $C > 0$, independent of n , such that*

$$\left(\int_{\mathcal{A}_n^2} y^{1-2\alpha} |v_n|^p dx dy \right)^{\frac{1}{p}} \leq C \sigma_n^{-\frac{N+2-2\alpha}{2p}} \quad (4.1)$$

and

$$\int_{\mathcal{A}_n^2 \cap \{y=0\}} |v_n|^p \leq C \sigma_n^{-\frac{N}{2}} \quad (4.2)$$

for any $p \geq 1$.

To prove Proposition 4.1, we need the following lemmas.

Lemma 4.1. *Let w_n be a solution of (3.4). There is a constant, independent of n , such that*

$$\frac{1}{r^{N+1-2\alpha}} \int_{\partial \mathcal{B}_r^+(z) \cap \{y>0\}} y^{1-2\alpha} w_n dS \leq C$$

for all $r \geq \bar{C} \sigma_n^{-1/2}$ and $z = (z', 0)$ with $z' \in \Omega$.

Proof. For $X = (x, y)$, $z = (z', 0)$,

$$\Gamma(X, z) = \frac{1}{|X - z|^{N-2\alpha}} - \frac{1}{s^{N-2\alpha}}$$

satisfies

$$\operatorname{div}(y^{1-2\alpha} \nabla_X \Gamma(X, z)) = 0 \quad \text{in } \mathcal{B}_s(z) \setminus \{z\}; \quad \Gamma(X, z) = 0, \quad X \in \partial \mathcal{B}_s(z),$$

where $\mathcal{B}_s(z) \subset \mathbb{R}^{N+1}$ is a ball centered at z with radius s .

Denote $f_n = 2|v_n|^{2^*_\alpha-1} + A$. Integrating by parts, we find that for $\delta \in (0, s)$,

$$\begin{aligned} 0 &= \int_{\mathcal{B}_s^+(z) \setminus \mathcal{B}_\delta^+(z)} \operatorname{div}(y^{1-2\alpha} \nabla w_n) \Gamma(X, z) dX \\ &= \int_{\partial(\mathcal{B}_s^+(z) \setminus \mathcal{B}_\delta^+(z))} y^{1-2\alpha} \frac{\partial w_n}{\partial n} \Gamma(X, z) dS - \int_{\partial(\mathcal{B}_s^+(z) \setminus \mathcal{B}_\delta^+(z))} y^{1-2\alpha} w_n \frac{\partial \Gamma}{\partial n} dS \\ &= \int_{\{y=0\} \cap (\mathcal{B}_s(z) \setminus \mathcal{B}_\delta(z))} f_n \Gamma(X, z) dX + \int_{\{y>0\} \cap \partial \mathcal{B}_\delta(z)} y^{1-2\alpha} \frac{\partial w_n}{\partial n} \Gamma(X, z) dS \\ &\quad - \int_{\{y>0\} \cap \partial(\mathcal{B}_s(z) \setminus \mathcal{B}_\delta(z))} y^{1-2\alpha} w_n \frac{\partial \Gamma}{\partial n} dS, \end{aligned} \quad (4.3)$$

since

$$\Gamma(X, z) = 0, \quad X \in \partial\mathcal{B}_s(z), \quad (4.4)$$

and

$$y^{1-2\alpha} \frac{\partial \Gamma(X, z)}{\partial n} = -\frac{(N-2\alpha)y^{2-2\alpha}}{|X-z|^{N-2\alpha+2}} = 0, \quad X \in \{y=0\} \cap (\mathcal{B}_s(z) \setminus \mathcal{B}_\delta(z)).$$

Differentiating (4.3) with respect to s , using (4.4), we are led to

$$\begin{aligned} & \int_{\{y=0\} \cap (\mathcal{B}_s(x) \setminus \mathcal{B}_\delta(z))} f_n \frac{N-2\alpha}{s^{N-2\alpha+1}} dX + \int_{\{y>0\} \cap \partial\mathcal{B}_\delta(z)} y^{1-2\alpha} \frac{\partial w_n}{\partial n} \frac{N-2\alpha}{s^{N-2\alpha+1}} dS \\ & + \frac{d}{ds} \int_{\{y>0\} \cap \partial\mathcal{B}_s(z)} y^{1-2\alpha} w_n \frac{N-2\alpha}{s^{N-2\alpha+1}} dS = 0. \end{aligned} \quad (4.5)$$

Letting $\delta \rightarrow 0$ in (4.5), we obtain the following formula

$$\frac{1}{s^{N-2\alpha+1}} \int_{\{y=0\} \cap \mathcal{B}_s(x)} f_n dX + \frac{d}{ds} \left(\frac{1}{s^{N-2\alpha+1}} \int_{\{y>0\} \cap \partial\mathcal{B}_s(z)} y^{1-2\alpha} w_n dS \right) = 0, \quad (4.6)$$

since

$$y^{1-2\alpha} \frac{\partial w_n}{\partial n} \rightarrow 2|v_n(x, 0)|^{2_\alpha^*-1} + A, \quad \text{as } y \rightarrow 0.$$

From

$$\begin{aligned} & \int_{\mathcal{B}_s(z) \cap \{y>0\}} y^{1-2\alpha} w_n dX \\ & \leq \left(\int_{\mathcal{B}_s(z) \cap \{y>0\}} y^{1-2\alpha} dX \right)^{\frac{1}{2}} \left(\int_{\mathcal{C}} y^{1-2\alpha} w_n^2 dX \right)^{\frac{1}{2}} \leq C, \end{aligned}$$

we can find a $r_n \in [\frac{1}{2}, 1]$, such that

$$\frac{1}{r_n^{N+1-2\alpha}} \int_{\partial\mathcal{B}_{r_n}(z) \cap \{y>0\}} y^{1-2\alpha} w_n dS \leq C.$$

Integrating (4.6) from r to r_n , we obtain

$$\begin{aligned}
& \frac{1}{r^{N+1-2\alpha}} \int_{\partial \mathcal{B}_r(z) \cap \{y>0\}} y^{1-2\alpha} w_n dS \\
&= \frac{1}{r^{N+1-2\alpha}} \int_{\partial \mathcal{B}_{r_n}(y) \cap \{y>0\}} y^{1-2\alpha} w_n dS + \int_r^{r_n} \frac{1}{t^{N+1-2\alpha}} \int_{\mathcal{B}_t(z) \cap \{y=0\}} f_n dS dt \\
&\leq C + \int_r^{r_n} \frac{1}{t^{N+1-2\alpha}} \int_{\{y=0\} \cap \mathcal{B}_t(z)} (2|v_n|^{2_\alpha^*-1} + A) dx dt \\
&\leq C + C \int_r^{r_n} \frac{1}{t^{N+1-2\alpha}} \int_{\{y=0\} \cap \mathcal{B}_t(y)} (w_n^{2_\alpha^*-1} + A) dx dt,
\end{aligned} \tag{4.7}$$

since $|v_n| \leq w_n$.

It is easy to check

$$\int_r^{r_n} \frac{1}{t^{N+1-2\alpha}} \int_{\{y=0\} \cap \mathcal{B}_t(z)} A dx dt \leq C \int_r^{r_n} t^{2\alpha-1} dt \leq C. \tag{4.8}$$

By Proposition 3.1, we know that $\|w_n(\cdot, 0)\|_{q_1, q_2, \sigma_n} \leq C$ for any $\frac{N}{N-2\alpha} < q_2 < 2_\alpha^* < q_1$. Let $q_1 > 2_\alpha^*$ large such that

$$-\frac{(N+2\alpha)}{q_1(N-2\alpha)} + 2\alpha - 1 > -1.$$

Let

$$q_2 = \frac{N+2\alpha}{N-2\alpha}.$$

Then, we can choose $v_{1,n}$, and $v_{2,n}$, such that $|w_n(x, 0)| \leq v_{1,n} + v_{2,n}$, and

$$\|v_{1,n}\|_{q_1} \leq C,$$

and

$$\|v_{2,n}\|_{q_2} \leq C \sigma_n^{\frac{N}{2_\alpha^*} - \frac{N}{q_2}}.$$

We have

$$\begin{aligned}
& \int_r^{r_n} \frac{1}{t^{N+1-2\alpha}} \int_{\{y=0\} \cap \mathcal{B}_t(z)} |v_{1,n}|^{2_\alpha^*-1} dx dt \\
& \leq \int_r^{r_n} \frac{1}{t^{N+1-2\alpha}} \left(\int_{\mathcal{B}_t(z) \cap \{y=0\}} |v_{1,n}|^{q_1} dx \right)^{\frac{N+2\alpha}{(N-2\alpha)q_1}} t^{N(1-\frac{N+2\alpha}{(N-2\alpha)q_1})} dt \\
& \leq C \int_r^1 t^{-\frac{(N+2\alpha)}{q_1(N-2\alpha)}+2\alpha-1} dt \leq C.
\end{aligned} \tag{4.9}$$

On the other hand, noting that $r \geq \bar{C}\sigma_n^{-1/2}$,

$$\begin{aligned}
& \int_r^{r_n} \frac{1}{t^{N+1-2\alpha}} \int_{\{y=0\} \cap \mathcal{B}_t(z)} |v_{2,n}|^{2_\alpha^*-1} dx dt \\
& \leq C \sigma_n^{(\frac{N}{2_\alpha^*} - \frac{N}{q_2})q_2} \int_r^{r_n} \frac{1}{t^{N+1-2\alpha}} dt \leq C \sigma_n^{(\frac{N}{2_\alpha^*} - \frac{N}{q_2})q_2} r^{2\alpha-N} \\
& \leq C \sigma_n^{(\frac{N}{2_\alpha^*} - \frac{N}{q_2})q_2 + \frac{N-2\alpha}{2}} = C.
\end{aligned} \tag{4.10}$$

Combining (4.8)-(4.10), we obtain

$$\int_r^{r_n} \frac{1}{t^{N+1-2\alpha}} \int_{\{y=0\} \cap \mathcal{B}_t(z)} (2|w_n|^{2_\alpha^*-1} + A) dx dt \leq C, \tag{4.11}$$

and then

$$\frac{1}{r^{N+1-2\alpha}} \int_{\partial \mathcal{B}_r^+(z) \cap \{y>0\}} y^{1-2\alpha} w_n dS \leq C.$$

□

Let us recall the Muckenhoupt class A_p for $p > 1$:

$$A_p = \left\{ w : \sup_{\mathcal{B}} \left(\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |w| \right) \left(\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |w|^{-\frac{1}{p-1}} \right)^{p-1} \leq C, \text{ for all ball } \mathcal{B} \text{ in } \mathbb{R}^{N+1} \right\}.$$

It is easy to check that $y^{1-2\alpha} \in A_2$.

Denote $\|u\|_{L^p(E, y^{1-2\alpha})} = \left(\int_E y^{1-2\alpha} |u|^p dx \right)^{\frac{1}{p}}$. We have the following result [15]:

Lemma 4.2. *Let \mathcal{D} be an open bounded set in \mathbb{R}^{N+1} . There exist constants $\delta > 0$ and $C > 0$ depending only on N and \mathcal{D} , such that for all $u \in C_0^\infty(\mathcal{D})$ and all k satisfying $1 \leq k \leq \frac{N}{N-1} + \delta$,*

$$\|u\|_{L^{2k}(\mathcal{D}, y^{1-2\alpha})} \leq C \|\nabla u\|_{L^2(\mathcal{D}, y^{1-2\alpha})}. \tag{4.12}$$

Let D^* be an open set in \mathbb{R}^N . Consider the following problem:

$$\begin{cases} \operatorname{div}(y^{1-2\alpha}\nabla w) = 0, & (x, y) \in \mathcal{C}_{D^*}; \\ -y^{1-2\alpha}\frac{\partial w}{\partial y} = a(x)w, & x \in D^*, y = 0, \end{cases} \quad (4.13)$$

where $a(x) \geq 0$ and $a \in L_{loc}^\infty(\mathbb{R}^N)$. We have the following estimate:

Lemma 4.3. *Suppose that w is a solution of (4.13). If there is a small constant $\delta > 0$ such that*

$$\int_{\mathcal{B}_1(z) \cap \{y=0\}} |a|^{\frac{N}{2\alpha}} dx \leq \delta,$$

for any $\mathcal{B}_1(z) \cap \{y = 0\} \subset D^*$, $z = (x, 0)$, then for any $p \geq 1$, there is a constant $C = C(p) > 0$ such that

$$\|w\|_{L^p(\mathcal{B}_{1/2}^+(z), y^{1-2\alpha})} \leq C \|w\|_{L^1(\mathcal{B}_1^+(z), y^{1-2\alpha})}, \quad (4.14)$$

and

$$\left(\int_{\mathcal{B}_r^+(z) \cap \{y=0\}} w^p dx \right)^{\frac{1}{p}} \leq \frac{C}{(R-r)^{\frac{\sigma}{\kappa}}} \|w\|_{L^1(\mathcal{B}_R^+(z), y^{1-2\alpha})} \quad (4.15)$$

for $p \geq 1$, $0 < \sigma \leq 1$ and $0 < \kappa < 1$.

Proof. We only need to prove the result for $p > 2_\alpha^*$. Let $1 \geq R > r > 0$. Define $\xi \in C_0^2(\mathcal{B}_R(z))$, with $\xi = 1$ in $\mathcal{B}_r(z)$, $0 \leq \xi \leq 1$, and $|\nabla \xi| \leq \frac{2}{R-r}$. Let $q = \frac{p}{2_\alpha^*}$ and R be small so that $\varphi = \xi^2 w^{2q-1} \in H_{0,L}^1(\mathcal{C}_{D^*})$. We have

$$\int_{\mathcal{C}_{D^*}} y^{1-2\alpha} \nabla w \nabla \varphi dx dy = \int_{D^* \cap \{y=0\}} a w \varphi dx,$$

and

$$\begin{aligned} & \int_{\mathcal{C}_{D^*}} y^{1-2\alpha} \nabla w \nabla \varphi dx dy \\ & \geq \frac{2q-1}{2q^2} \int_{\mathcal{C}_{D^*}} y^{1-2\alpha} |\nabla(\xi w^q)|^2 dx dy - \frac{C}{(R-r)^2} \int_{\mathcal{B}_R^+(z)} y^{1-2\alpha} w^{2q} dx. \end{aligned}$$

Hence,

$$\begin{aligned}
& \int_{\mathcal{C}_{D^*}} y^{1-2\alpha} |\nabla(\xi w^q)|^2 dx dy \\
& \leq \frac{C}{(R-r)^2} \int_{\mathcal{B}_R^+(z)} y^{1-2\alpha} w^{2q} dx + \int_{D^* \times \{0\}} a w \varphi dx \\
& \leq \frac{C}{(R-r)^2} \int_{\mathcal{B}_R^+(z)} y^{1-2\alpha} w^{2q} dx \\
& \quad + \left(\int_{\mathcal{B}_R^+(z) \cap \{y=0\}} |a|^{\frac{N}{2\alpha}} dx \right)^{\frac{2\alpha}{N}} \left(\int_{D^* \times \{0\}} (\xi w^q)^{2^*} dx \right)^{\frac{N-2\alpha}{N}} \\
& \leq \frac{C}{(R-r)^2} \int_{\mathcal{B}_R^+(z)} y^{1-2\alpha} w^{2q} dx + \delta^{\frac{2\alpha}{N}} \left(\int_{D^* \times \{0\}} (\xi w^q)^{2^*} dx \right)^{\frac{N-2\alpha}{N}}.
\end{aligned} \tag{4.16}$$

By the trace inequality, we obtain

$$\begin{aligned}
& \int_{\mathcal{C}_{D^*}} y^{1-2\alpha} |\nabla(\xi w^q)|^2 dx dy \\
& \leq \frac{C}{(R-r)^2} \int_{\mathcal{B}_R^+(z)} y^{1-2\alpha} w^{2q} dx dy + C \delta^{\frac{2\alpha}{N}} \int_{\mathcal{C}_{D^*}} y^{1-2\alpha} |\nabla(\xi w^q)|^2 dx dy.
\end{aligned} \tag{4.17}$$

So, if $\delta > 0$ is small, we obtain

$$\int_{\mathcal{C}_{D^*}} y^{1-2\alpha} |\nabla(\xi w^q)|^2 dx dy \leq \frac{C}{(R-r)^2} \int_{\mathcal{B}_R^+(z)} y^{1-2\alpha} w^{2q} dx dy \tag{4.18}$$

for $0 < r < R < 1$. By Lemma 4.2,

$$\left(\int_{\mathcal{B}_R^+(z)} y^{1-2\alpha} (\xi w^q)^{2t} dx \right)^{\frac{1}{t}} \leq C \int_{\mathcal{B}_R^+(z)} y^{1-2\alpha} |\nabla(\xi w^q)|^2 dx dy \tag{4.19}$$

for some $t > 1$. As a result, we obtain from (4.18) and (4.19),

$$\left(\int_{\mathcal{B}_r^+(z)} y^{1-2\alpha} w^{2tq} dx \right)^{\frac{1}{t}} \leq \frac{C}{(R-r)^2} \int_{\mathcal{B}_R^+(z)} y^{1-2\alpha} w^{2q} dx dy, \tag{4.20}$$

which yields

$$\left(\int_{\mathcal{B}_r^+(z)} y^{1-2\alpha} w^{2tq} dx \right)^{\frac{1}{2tq}} \leq \frac{C}{(R-r)^{\frac{1}{q}}} \left(\int_{\mathcal{B}_R^+(z)} y^{1-2\alpha} w^{2q} dx dy \right)^{\frac{1}{2q}} \quad (4.21)$$

for $0 < r < R < 1$. Note that if $p > q \geq 1$, by Hölder's inequality,

$$\left(\int_{\mathcal{B}_R^+(z)} y^{1-2\alpha} w^q dx \right)^{\frac{1}{q}} \leq C \left(\int_{\mathcal{B}_R^+(z)} y^{1-2\alpha} w^p dx dy \right)^{\frac{1}{p}}. \quad (4.22)$$

Using (4.22) and iterating (4.21) we obtain that there is $\sigma > 0$ such that

$$\left(\int_{\mathcal{B}_r^+(z)} y^{1-2\alpha} w^p dx \right)^{\frac{1}{p}} \leq \frac{C}{(R-r)^\sigma} \left(\int_{\mathcal{B}_R^+(z)} y^{1-2\alpha} w^{2_\alpha^*} dx dy \right)^{\frac{1}{2_\alpha^*}} \quad (4.23)$$

for $p > 2_\alpha^*$ and $0 < r < R < 1$. By Hölder's inequality,

$$\left(\int_{\mathcal{B}_R^+(z)} y^{1-2\alpha} w^{2_\alpha^*} dx dy \right)^{\frac{1}{2_\alpha^*}} \leq \|w\|_{L^1(\mathcal{B}_R^+(z), y^{1-2\alpha})}^\kappa \|w\|_{L^p(\mathcal{B}_R^+(z), y^{1-2\alpha})}^{1-\kappa}.$$

Hence,

$$\|w\|_{L^p(\mathcal{B}_r^+(z), y^{1-2\alpha})} \leq \frac{1}{2} \|w\|_{L^p(\mathcal{B}_R^+(z), y^{1-2\alpha})} + \frac{C}{(R-r)^{\frac{\sigma}{\kappa}}} \|w\|_{L^1(\mathcal{B}_R^+(z), y^{1-2\alpha})}.$$

By iteration, we obtain

$$\|w\|_{L^p(\mathcal{B}_r^+(z), y^{1-2\alpha})} \leq \frac{C}{(R-r)^{\frac{\sigma}{\kappa}}} \|w\|_{L^1(\mathcal{B}_R^+(z), y^{1-2\alpha})} \quad (4.24)$$

for $p > 2_\alpha^*$ and $0 < r < R < 1$.

Finally, (4.18), (4.24) and the trace inequality imply that

$$\left(\int_{\mathcal{B}_r^+(z) \cap \{y=0\}} w^p dx \right)^{\frac{1}{p}} \leq \frac{C}{(R-r)^{\frac{\sigma}{\kappa}}} \|w\|_{L^1(\mathcal{B}_R^+(z), y^{1-2\alpha})}. \quad (4.25)$$

□

Proof of Proposition 4.1. It follows from Lemma 4.1

$$\frac{1}{r^{N+1-2\alpha}} \int_{\partial \mathcal{B}_r^+((x_n, 0))} y^{1-2\alpha} w_n dS \leq C,$$

which gives

$$\int_{\mathcal{A}_n^2} y^{1-2\alpha} w_n dX \leq C \int_{(C+1)\sigma_n^{-\frac{1}{2}}}^{(C+4)\sigma_n^{-\frac{1}{2}}} r^{N+1-2\alpha} dr \leq C \sigma_n^{-\frac{N+2-2\alpha}{2}}.$$

In particular

$$\int_{\mathcal{B}_{\sigma_n^{-\frac{1}{2}}}(z)} y^{1-2\alpha} w_n dX \leq C \sigma_n^{-\frac{N+2-2\alpha}{2}}, \quad \forall z \in \mathcal{A}_n^2. \quad (4.26)$$

Let

$$\tilde{v}_n(X) = v_n(\sigma_n^{-\frac{1}{2}} X), \quad X = (x, y) \in \mathcal{C}_{\Omega_n},$$

where $\Omega_n = \{x : \sigma_n^{-\frac{1}{2}} x \in \Omega\}$. Then \tilde{v}_n satisfies

$$\begin{cases} \operatorname{div}(y^{1-2\alpha} \nabla \tilde{v}_n) = 0, & \text{in } \mathcal{C}_{\Omega_n}, \\ \tilde{v}_n = 0, & \text{on } \partial_L \mathcal{C}_{\Omega_n}, \\ y^{1-2\alpha} \frac{\partial \tilde{v}_n}{\partial \nu} = \sigma_n^{-\alpha} (|\tilde{v}_n(x, 0)|^{p_n-2} \tilde{v}_n(x, 0) + \lambda \tilde{v}_n(x, 0)), & \text{on } \Omega_n \times \{0\}. \end{cases}$$

Let $\xi = \sigma_n^{\frac{1}{2}} z$. Since $B_{\sigma_n^{-\frac{1}{2}}}(z)$, $z \in \mathcal{A}_n^2$, does not contain any concentration point of v_n , we can deduce

$$\begin{aligned} & \int_{\mathcal{B}_1(\xi) \cap \{y=0\}} |\sigma_n^{-\alpha} (|\tilde{v}_n(x, 0)|^{p_n-2} + \lambda)|^{\frac{N}{2\alpha}} dx \\ & \leq C \int_{\mathcal{B}_1(\xi) \cap \{y=0\}} |\sigma_n^{-\alpha} (|\tilde{v}_n(x, 0)|^{2_\alpha^*-2} + 1)|^{\frac{N}{2\alpha}} dx \\ & \leq C \int_{\mathcal{B}_{\sigma_n^{-\frac{1}{2}}}(z) \cap \{y=0\}} |v_n|^{2_\alpha^*} dx + C \sigma_n^{-\frac{N}{2}} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Thus, by Lemmas 4.1 and 4.3, noting $|v_n| \leq w_n$, we obtain

$$\begin{aligned}
\|\tilde{v}_n\|_{L^p(\mathcal{B}_{\frac{1}{2}}^+(\xi), y^{1-2\alpha})} &\leq C \int_{\mathcal{B}_1^+(\xi)} y^{1-2\alpha} |\tilde{v}_n| \, dx dy \\
&\leq C \sigma_n^{\frac{N+2-2\alpha}{2}} \int_{\mathcal{B}_{\frac{1}{2}\sigma_n}^+(z)} y^{1-2\alpha} |w_n| \, dx dy \\
&\leq C \sigma_n^{\frac{N+2-2\alpha}{2}} \int_{\mathcal{A}_n^2} y^{1-2\alpha} |w_n| \, dx dy \\
&\leq C \sigma_n^{\frac{N+2-2\alpha}{2}} \sigma_n^{-\frac{N+2-2\alpha}{2}} \leq C.
\end{aligned}$$

By (4.25), we also have

$$\left(\int_{\mathcal{B}_{\frac{1}{2}}^+(\xi) \cap \{y=0\}} |\tilde{v}_n|^p \, dx \right)^{\frac{1}{p}} \leq C \|\tilde{v}_n\|_{L^1(\mathcal{B}_1^+(\xi), y^{1-2\alpha})} \leq C.$$

As a result,

$$\sigma_n^{\frac{N+2-2\alpha}{2p}} \left(\int_{\mathcal{B}_{\frac{1}{2}\sigma_n}^+(z)} y^{1-2\alpha} |v_n|^p \, dx dy \right)^{\frac{1}{p}} \leq C, \quad \forall z \in \mathcal{A}_n^2.$$

Thus,

$$\int_{\mathcal{A}_n^2} y^{1-2\alpha} |v_n|^p \leq C \sigma_n^{-\frac{N+2-2\alpha}{2}}.$$

Similarly,

$$\int_{\mathcal{A}_n^2 \cap \{y=0\}} |v_n|^p \leq C \sigma_n^{-\frac{N}{2}}.$$

□

Proposition 4.2. *We have*

$$\begin{aligned}
&\int_{\mathcal{A}_n^3} y^{1-2\alpha} |\nabla v_n|^2 \, dx dy \\
&\leq C \sigma_n \int_{\mathcal{A}_n^2} y^{1-2\alpha} |w_n|^2 \, dx dy + C \int_{\mathcal{A}_n^2 \times \{y=0\}} |w_n|^{2^*} \, dx + C \int_{\mathcal{A}_n^2 \times \{y=0\}} |w_n|^2 \, dx.
\end{aligned} \tag{4.27}$$

In particular,

$$\int_{\mathcal{A}_n^3} y^{1-2\alpha} |\nabla v_n|^2 dx dy \leq C \sigma_n^{-\frac{N-2\alpha}{2}}. \quad (4.28)$$

Proof. Let $\varphi_n \in C_0^2(\mathcal{A}_n^2)$ be a function with $\varphi_n = 1$ in \mathcal{A}_n^3 ; $0 \leq \varphi_n \leq 1$ and $|\nabla \varphi_n| \leq C \sigma_n^{\frac{1}{2}}$. From

$$\begin{aligned} \int_{\mathcal{C}_\Omega} y^{1-2\alpha} \nabla v_n \nabla (\varphi_n^2 v_n) dx dy &= \int_{\Omega \times \{y=0\}} (|v_n|^{2_\alpha^*-2} + \lambda) v_n \varphi_n^2 v_n dx \\ &\leq C \int_{\Omega \times \{y=0\}} (|w_n|^{2_\alpha^*} + w_n^2) \varphi_n^2 dx, \end{aligned}$$

we can prove (4.27).

On the other hand, it follows from Proposition 4.1 that

$$\begin{aligned} \sigma_n \int_{\mathcal{A}_n^2} y^{1-2\alpha} |w_n|^2 dx dy + \int_{\mathcal{A}_n^2 \times \{y=0\}} |w_n|^{2_\alpha^*} dx + \int_{\mathcal{A}_n^2 \times \{y=0\}} |w_n|^2 dx \\ \leq C \sigma_n \sigma_n^{-\frac{N+2-2\alpha}{2}} + C \sigma_n^{-\frac{N}{2}} \leq C \sigma_n^{-\frac{N-2\alpha}{2}}. \end{aligned} \quad (4.29)$$

It yields from (4.27) that

$$\int_{\mathcal{A}_n^3} y^{1-2\alpha} |\nabla v_n|^2 dx dy \leq C \sigma_n^{-\frac{N-2\alpha}{2}}.$$

□

5. EXISTENCE OF INFINITELY MANY BOUND STATE SOLUTIONS

Firstly, we have the following local Pohozaev identity.

Lemma 5.1. *Let v be a solution of (1.5). Then for any smooth subset $\mathcal{M} \subset \mathcal{C}_\Omega$, v satisfies*

$$\begin{aligned} \frac{N-2\alpha}{2} \int_{\partial \mathcal{M}} y^{1-2\alpha} v \frac{\partial v}{\partial \nu} dS \\ = \frac{1}{2} \int_{\partial \mathcal{M}} y^{1-2\alpha} |\nabla v|^2 (X - z_0, \nu) dS - \int_{\partial \mathcal{M}} y^{1-2\alpha} (\nabla v, X - z_0) \frac{\partial v}{\partial \nu} dS, \end{aligned} \quad (5.1)$$

where ν is the outward normal to ∂S , and $z_0 \in \mathbb{R}^{N+1}$.

Proof of Theorem 1.1 We argue by contradiction. Suppose the assertion is not true. Choose $t_n \in [\bar{C} + 2, \bar{C} + 3]$ so that

$$\begin{aligned}
& \int_{\left(\partial \mathcal{B}_{t_n \sigma_n^{-\frac{1}{2}}}((x_n, 0))\right) \cap \mathcal{C}_\Omega} y^{1-2\alpha} \left(\sigma_n^{-\frac{1}{2}} |\nabla v_n|^2 + \sigma_n^{\frac{1}{2}} v_n^2 \right) dS \\
& + \sigma_n^{\alpha-\frac{1}{2}} \int_{\left(\partial \mathcal{B}_{t_n \sigma_n^{-\frac{1}{2}}}((x_n, 0))\right) \cap (\Omega \times \{0\})} \left(|v_n|^{2_\alpha^*} + v_n^2 \right) dS \\
& \leq \int_{\mathcal{A}_n^3} y^{1-2\alpha} \left(|\nabla v_n|^2 + \sigma_n v_n^2 \right) dx dy + \sigma_n^\alpha \int_{\mathcal{A}_n^3 \cap \{y=0\}} \left(|v_n|^{2_\alpha^*} + v_n^2 \right) dx,
\end{aligned} \tag{5.2}$$

By Propositions 4.1 and 4.2,

$$\begin{aligned}
& \int_{\left(\partial \mathcal{B}_{t_n \sigma_n^{-\frac{1}{2}}}((x_n, 0))\right) \cap \mathcal{C}_\Omega} y^{1-2\alpha} \left(\sigma_n^{-\frac{1}{2}} |\nabla v_n|^2 + \sigma_n^{\frac{1}{2}} v_n^2 \right) dS \\
& + \sigma_n^{\alpha-\frac{1}{2}} \int_{\left(\partial \mathcal{B}_{t_n \sigma_n^{-\frac{1}{2}}}((x_n, 0))\right) \cap (\Omega \times \{0\})} \left(|v_n|^{2_\alpha^*} + v_n^2 \right) dS \\
& \leq C \sigma_n^{-\frac{N-2\alpha}{2}} + C \sigma_n^\alpha \sigma_n^{-\frac{N}{2}} = C' \sigma_n^{-\frac{N-2\alpha}{2}}.
\end{aligned} \tag{5.3}$$

Let $p_n = 2_\alpha^* - \varepsilon_n$. Applying Lemma 5.1 to v_n on $\mathcal{B}_n = \mathcal{B}_{t_n \sigma_n^{-\frac{1}{2}}}((x_n, 0)) \cap \mathcal{C}_\Omega \subset \mathbb{R}^{N+1}$ and $z_0 = (x_0, 0)$, we obtain

$$\begin{aligned}
& \frac{N-2\alpha}{2} \int_{\partial \mathcal{B}_n} y^{1-2\alpha} v_n \frac{\partial v_n}{\partial \nu} dS \\
& = \frac{1}{2} \int_{\partial \mathcal{B}_n} y^{1-2\alpha} |\nabla v_n|^2 (X - z_0, \nu) dS - \int_{\partial \mathcal{B}_n} y^{1-2\alpha} (\nabla v_n, X - z_0) \frac{\partial v_n}{\partial \nu} dS.
\end{aligned} \tag{5.4}$$

From the fact that

$$y^{1-2\alpha} v_n \frac{\partial v_n}{\partial \nu} = |v_n(x, 0)|^{p_n-2} v_n(x, 0) + \lambda v_n(x, 0), \quad \text{on } y = 0,$$

we obtain from (5.4)

$$\begin{aligned}
& \frac{N-2\alpha}{2} \int_{\mathcal{B}_n \cap \{y=0\}} (|v_n|^{p_n} + \lambda v_n^2) dx + \frac{N-2\alpha}{2} \int_{\partial \mathcal{B}_n \cap \{y>0\}} y^{1-2\alpha} v_n \frac{\partial v_n}{\partial \nu} dS \\
&= \frac{1}{2} \int_{\mathcal{B}_n \cap \{y=0\}} y^{1-2\alpha} |\nabla v_n|^2 (x - x_0, \nu) dx + \frac{1}{2} \int_{\partial \mathcal{B}_n \cap \{y>0\}} y^{1-2\alpha} |\nabla v_n|^2 (X - z_0, \nu) dS \\
&- \int_{\mathcal{B}_n \cap \{y=0\}} y^{1-2\alpha} (\nabla v_n, x - x_0) \frac{\partial v_n}{\partial \nu} dx - \int_{\partial \mathcal{B}_n \cap \{y>0\}} y^{1-2\alpha} (\nabla v_n, X - z_0) \frac{\partial v_n}{\partial \nu} dS.
\end{aligned} \tag{5.5}$$

Noting that $x - x_0 \perp \nu$ on $\mathcal{B}_n \cap \{y = 0\}$, we find

$$\int_{\mathcal{B}_n \cap \{y=0\}} y^{1-2\alpha} |\nabla v_n|^2 (x - x_0, \nu) dx = 0.$$

On the other hand,

$$\begin{aligned}
& - \int_{\mathcal{B}_n \cap \{y=0\}} y^{1-2\alpha} (\nabla v_n, x - x_0) \frac{\partial v_n}{\partial \nu} dx \\
&= - \int_{\mathcal{B}_n \cap \{y=0\}} (\nabla_x v_n, x - x_0) (|v_n|^{p_n-2} v_n + \lambda v_n) dx. \\
&= - \int_{\mathcal{B}_n \cap \{y=0\}} (\nabla_x (\frac{1}{p_n} |v_n|^{p_n} + \frac{1}{2} \lambda v_n^2), x - x_0) dx \\
&= N \int_{\mathcal{B}_n \cap \{y=0\}} (\frac{1}{p_n} |v_n|^{p_n} + \frac{1}{2} \lambda v_n^2) dx - \int_{\partial(\mathcal{B}_n \cap \{y=0\})} (\frac{1}{p_n} |v_n|^{p_n} + \frac{1}{2} \lambda v_n^2) \langle x - x_0, \nu_x \rangle dS.
\end{aligned} \tag{5.6}$$

So equation (5.5) becomes

$$\begin{aligned}
& \left(\frac{N}{p_n} - \frac{N-2\alpha}{2} \right) \int_{\mathcal{B}_n \cap \{y=0\}} |v_n|^{p_n} dx + \left(\frac{N}{2} - \frac{N-2\alpha}{2} \right) \lambda \int_{\mathcal{B}_n \cap \{y=0\}} v_n^2 dx \\
&= \int_{\partial(\mathcal{B}_n \cap \{y=0\})} \left(\frac{1}{p_n} |v_n|^{p_n} + \frac{1}{2} \lambda v_n^2 \right) \langle x - x_0, \nu_x \rangle dS \\
&+ \frac{N-2\alpha}{2} \int_{\partial \mathcal{B}_n \cap \{y>0\}} y^{1-2\alpha} v_n \frac{\partial v_n}{\partial \nu} dS \\
&- \frac{1}{2} \int_{\partial \mathcal{B}_n \cap \{y>0\}} y^{1-2\alpha} |\nabla v_n|^2 (X - z_0, \nu) dS \\
&+ \int_{\partial \mathcal{B}_n \cap \{y>0\}} y^{1-2\alpha} (\nabla v_n, X - z_0) \frac{\partial v_n}{\partial \nu} dS.
\end{aligned} \tag{5.7}$$

We decompose

$$\partial \mathcal{B}_n \cap \{y > 0\} = \partial_i \mathcal{B}_n \cup \partial_e \mathcal{B}_n,$$

where $\partial_i \mathcal{B}_n = \partial \mathcal{B}_n \cap \mathcal{C}_\Omega$ and $\partial_e \mathcal{B}_n = \mathcal{B}_n \cap \partial_L \mathcal{C}_\Omega$.

Now, we have two cases:

- (i) $\mathcal{B}_{t_n \sigma_n^{-\frac{1}{2}}}((x_n, 0)) \cap \{y > 0\} \cap (\mathbb{R}^{N+1} \setminus \mathcal{C}_\Omega) \neq \emptyset$,
- (ii) $\mathcal{B}_{t_n \sigma_n^{-\frac{1}{2}}}((x_n, 0)) \cap \{y > 0\} \subset \mathcal{C}_\Omega$.

In case (i), we take $x_0 \in \mathbb{R}^N \setminus \Omega$ with $|x_n - x_0| \leq 2t_n \sigma_n^{-\frac{1}{2}}$, $\nu \cdot (X - (x_0, 0)) \leq 0$ on $\partial_e \mathcal{B}_n$, where ν is the outward normal to $\partial_L \mathcal{C}_\Omega$. Since $v_n = 0$ on $\partial_L \mathcal{C}_\Omega$, we find

$$\begin{aligned}
& - \frac{1}{2} \int_{\partial_e \mathcal{B}_n} y^{1-2\alpha} |\nabla v_n|^2 (X - z_0, \nu) dS \\
& + \int_{\partial_e \mathcal{B}_n} y^{1-2\alpha} (\nabla v_n, X - z_0) \frac{\partial v_n}{\partial \nu} dS. \\
&= \frac{1}{2} \int_{\partial_e \mathcal{B}_n} y^{1-2\alpha} |\nabla v_n|^2 (X - z_0, \nu) dS \leq 0.
\end{aligned} \tag{5.8}$$

In case (ii), $\partial_e \mathcal{B}_n = \emptyset$. We choose $x_0 = x_n$.

Noting that $p_n \leq 2_\alpha^*$ and $v_n = 0$ on $\partial_L \mathcal{C}_\Omega$, we obtain from (5.7)

$$\begin{aligned}
& \left(\frac{N}{2} - \frac{N-2\alpha}{2} \right) \lambda \int_{\mathcal{B}_n \cap \{y=0\}} v_n^2 dx \\
& \leq \int_{(\partial_i \mathcal{B}_n) \cap \{y=0\}} \left(\frac{1}{p_n} |v_n|^{p_n} + \frac{1}{2} \lambda v_n^2 \right) \langle x - x_0, \nu_x \rangle dS \\
& + \frac{N-2\alpha}{2} \int_{\partial_i \mathcal{B}_n} y^{1-2\alpha} v_n \frac{\partial v_n}{\partial \nu} dS \\
& - \frac{1}{2} \int_{\partial_i \mathcal{B}_n} y^{1-2\alpha} |\nabla v_n|^2 (X - z_0, \nu) dS \\
& + \int_{\partial_i \mathcal{B}_n} y^{1-2\alpha} (\nabla v_n, X - z_0) \frac{\partial v_n}{\partial \nu} dS.
\end{aligned} \tag{5.9}$$

By (5.3), we find

$$\begin{aligned}
\text{RHS of (5.9)} & \leq C \sigma_n^{-\frac{1}{2}} \int_{(\partial_i \mathcal{B}_n) \cap \{y=0\}} (|v_n|^{p_n} + v_n^2) dS \\
& + C \left(\int_{\partial_i \mathcal{B}_n} y^{1-2\alpha} |\nabla v_n|^2 dS \right)^{\frac{1}{2}} \left(\int_{\partial_i \mathcal{B}_n} y^{1-2\alpha} v_n^2 dS \right)^{\frac{1}{2}} \\
& + C \sigma_n^{-\frac{1}{2}} \int_{\partial_i \mathcal{B}_n} y^{1-2\alpha} |\nabla v_n|^2 dS \\
& \leq C \left(\sigma_n^{-\frac{1}{2}} \sigma_n^{\frac{1}{2}-\alpha} + \sigma_n^{\frac{1}{4}} \sigma_n^{-\frac{1}{4}} + \sigma_n^{-\frac{1}{2}} \sigma_n^{\frac{1}{2}} \right) \sigma_n^{-\frac{N-2\alpha}{2}} \leq C \sigma_n^{-\frac{N-2\alpha}{2}}.
\end{aligned} \tag{5.10}$$

Inserting (5.10) into (5.9), we obtain

$$\int_{\mathcal{B}_n \cap \{y=0\}} v_n^2 dx \leq C \sigma_n^{-\frac{N-2\alpha}{2}}. \tag{5.11}$$

Let us assume that $\sigma_n = \sigma_{n,1}$. Using (2.3), similarly to [12], we can deduce that if $N > 4\alpha$, then

$$\begin{aligned}
\int_{\mathcal{B}_n \cap \{y=0\}} v_n^2 dx &\geq \int_{\mathcal{B}_{\sigma_n^{-1}}((x_n, 0)) \cap \{y=0\}} v_n^2 dx \\
&\geq \frac{1}{2} \int_{\mathcal{B}_{\sigma_n^{-1}}((x_n, 0)) \cap \{y=0\}} |\rho_{x_n^1, \sigma_n^1}(W_1)|^2 + o(\sigma_n^{-2\alpha}) \\
&= \frac{1}{2} \sigma_n^{-2\alpha} \int_{\mathcal{B}_1(0) \cap \{y=0\}} W_1^2 + o(\sigma_n^{-2\alpha}).
\end{aligned} \tag{5.12}$$

Combining (5.11) and (5.12), we are led to

$$\sigma_n^{-2\alpha} \leq C \sigma_n^{-\frac{N-2\alpha}{2}}.$$

This is a contradiction if $N > 6\alpha$. \square

Proof of Theorem 1.2. It is standard to prove that Theorem 1.2 follows directly from Theorem 1.1. See [9, 12]. For the convenience of the readers, we follow [12] to outline the proof.

For any $k \in \mathbb{N}$, define the \mathbb{Z}_2 -homotopy class \mathcal{F}_k by

$$\mathcal{F}_k = \{A; A \in H_{0,L}^1(\mathcal{C}_\Omega) \text{ is compact, } \mathbb{Z}_2 - \text{invariant, and } \gamma(A) \geq k\},$$

where the genus $\gamma(A)$ is smallest integer m , such that there exists an odd map $\phi \in C(A, \mathbb{R}^m \setminus \{0\})$. For $k = 1, 2, \dots$, we can define the minimax value

$$c_{k,\varepsilon} = \inf_{A \in \mathcal{F}_k} \max_{u \in A} I_\varepsilon(u), \tag{5.13}$$

where $I_\varepsilon(u)$ is defined in (1.6). Then, $c_{k,\varepsilon}$ is a critical value of $I_\varepsilon(u)$. Thus there is $u_{k,\varepsilon}$ such that $I_\varepsilon(u_{k,\varepsilon}) = c_{k,\varepsilon}$ and $I'_\varepsilon(u_{k,\varepsilon}) = 0$.

For any $k = 1, \dots$, it is easy to show that $|c_{k,\varepsilon}| \leq C_k$ for some $C_k > 0$ which is independent of ε . Therefore, $u_{k,\varepsilon}$ is bounded in $H_{0,L}^1(\mathcal{C}_\Omega)$ for any fixed k . By Theorem 1.1, up to a subsequence, $u_{k,\varepsilon} \rightarrow u_k$ strongly in $H_{0,L}^1(\mathcal{C}_\Omega)$. So, u_k satisfies $I_0(u_k) = c_k := \lim_{\varepsilon \rightarrow 0} c_{k,\varepsilon}$ and $I'_0(u_k) = 0$.

We are now ready to show that $I_0(u)$ has infinitely many critical points. Note that c_k is non-decreasing in k . We distinguish several cases.

(1) Suppose that there are $1 < k_1 < \dots < k_i < \dots$, satisfying

$$c_{k_1} < \dots < c_{k_i} < \dots.$$

Then, we are done. So we assume in the sequel that for some positive integer m , $c_k = c$ for all $k \geq m$.

(2) Suppose that for any $\delta > 0$, $I_0(u)$ has a critical point u with $I_0(u) \in (c - \delta, c + \delta)$ and $I_0(u) \neq c$. In this case, we are done. So from now on we assume that there exists a $\delta > 0$, such that $I_0(u)$ has no critical point u with $I_0(u) \in (c - \delta, c) \cup (c, c + \delta)$. In this case, using the deformation argument, we can prove that

$$\gamma(K_c) \geq 2, \quad (5.14)$$

where $K_c = \{u \in H_{0,L}^1(\mathcal{C}_\Omega) : I_0'(u) = 0, I_0(u) = c\}$. As a consequence, $I_0(u)$ has infinitely many critical points. \square

APPENDIX A. ESTIMATES FOR A LINEAR PROBLEM

In this section, we will establish the L^p estimates for a linear problem. Let D be any bounded domain in \mathbb{R}^N . Recall that we use the notations $\mathcal{C}_D = D \times (0, +\infty)$ and $\partial_L \mathcal{C}_D = \partial D \times (0, +\infty)$. Consider

$$\begin{cases} \operatorname{div}(y^{1-2\alpha} \nabla w) = 0 & \text{in } \mathcal{C}_D, \\ w = 0 & \text{on } \partial_L \mathcal{C}_D, \\ y^{1-2\alpha} \frac{\partial w}{\partial \nu} = f(x) & \text{on } D \times \{0\}. \end{cases} \quad (\text{A.1})$$

Proposition A.1. *Suppose that $f \in C^\beta(D)$, $f \geq 0$. Let w be the solution of (A.1). Then for any $1 < p < \frac{N}{2\alpha}$, there is a constant $C > 0$, such that*

$$\|w(\cdot, 0)\|_{L^{\frac{Np}{N-2\alpha p}}(D)} \leq C \|f\|_{L^p(D)}.$$

Proof. First, it is easy to see that $w > 0$.

We claim that if $q > \frac{1}{2}$, then

$$\left(\int_D |w^q(x, 0)|^{2_\alpha^*} dx \right)^{2/2_\alpha^*} \leq C \int_D f(x) w^{2q-1}(x, 0) dx. \quad (\text{A.2})$$

Note that $w \in L^\infty(\mathcal{C}_D)$.

We first assume $q \geq 1$. Let $\varphi = w^{2q-1} \in H_{0,L}^1(\mathcal{C}_D)$. Testing (A.1) by φ , we obtain

$$\int_{\mathcal{C}_D} y^{1-2\alpha} \nabla w \nabla \varphi \, dx dy = \int_D f(x) w^{2q-1}(x, 0) \, dx.$$

We deduce

$$\begin{aligned}
& \int_{\mathcal{C}_D} y^{1-2\alpha} \nabla w \nabla \varphi \, dx dy \\
&= \frac{2q-1}{q^2} \int_{\mathcal{C}_D} y^{1-2\alpha} |\nabla w^q|^2 \, dx dy \\
&\geq c_0(q) \left(\int_D |w^q(x, 0)|^{2_\alpha^*} \, dx \right)^{2/2_\alpha^*}
\end{aligned}$$

where $c_0(q) > 0$ is some constant. Hence,

$$\left(\int_D |w^q(x, 0)|^{2_\alpha^*} \, dx \right)^{2/2_\alpha^*} \leq C \int_D f(x) w^{2q-1}(x, 0) \, dx. \quad (\text{A.3})$$

Now we consider the case $q \in (\frac{1}{2}, 1)$. For any $\theta > 0$, let $\eta = w(w + \theta)^{2(q-1)} \in H_{0,L}^1(\mathcal{C}_D)$. Then

$$\nabla \eta = (w + \theta)^{2(q-1)} \nabla w + 2(q-1)w(w + \theta)^{2q-3} \nabla w$$

From $q \in (\frac{1}{2}, 1)$, we find

$$\begin{aligned}
& \int_{\mathcal{C}_D} y^{1-2\alpha} \nabla w \nabla \eta \\
&\geq (2q-1) \int_{\mathcal{C}_D} y^{1-2\alpha} (w + \theta)^{2(q-1)} |\nabla w|^2 \\
&= \frac{2q-1}{q^2} \int_{\mathcal{C}_D} y^{1-2\alpha} |\nabla((w + \theta)^q - \theta^q)|^2 \\
&\geq c_0(q) \left(\int_D |(w(x, 0) + \theta)^q - \theta^q|^{2_\alpha^*} \, dx \right)^{2/2_\alpha^*}.
\end{aligned}$$

So, we obtain

$$\left(\int_D |(w(x, 0) + \theta)^q - \theta^q|^{2_\alpha^*} \, dx \right)^{2/2_\alpha^*} \leq C \int_D f(x) w(x, 0) (w(x, 0) + \theta)^{2q-2} \, dx. \quad (\text{A.4})$$

Letting $\theta \rightarrow 0$ in (A.4), we obtain (A.2).

On the other hand,

$$\int_D f(x) w^{2q-1}(x, 0) dx \leq \left(\int_D |f|^{\frac{2_\alpha^* q}{2_\alpha^* q - 2q + 1}} \right)^{\frac{2_\alpha^* q - 2q + 1}{2_\alpha^* q}} \|w\|_{q 2_\alpha^*}^{2q-1}, \quad (\text{A.5})$$

By (A.3), (A.5) and the embedding $H_{0,L}^1(\mathcal{C}_D) \hookrightarrow L^{2_\alpha^*}(\Omega)$, which, together with (A.2), gives

$$\|w\|_{q 2_\alpha^*} \leq C \left(\int_D |f|^{\frac{2_\alpha^* q}{2_\alpha^* q - 2q + 1}} dx \right)^{\frac{2_\alpha^* q - 2q + 1}{2_\alpha^* q}}. \quad (\text{A.6})$$

Let $p = \frac{2_\alpha^* q}{2_\alpha^* q - 2q + 1}$. Then $q = \frac{p}{2_\alpha^* - (2_\alpha^* - 2)p} > \frac{1}{2}$, and

$$2_\alpha^* q = \frac{2_\alpha^* p}{2_\alpha^* - (2_\alpha^* - 2)p} = \frac{Np}{N - 2p\alpha}.$$

The proof is complete. \square

Let $w \in H_{0,L}^1(\mathcal{C}_D)$ be a solution of

$$\begin{cases} \operatorname{div}(y^{1-2\alpha} \nabla w) = 0 & \text{in } \mathcal{C}_D, \\ w = 0 & \text{on } \partial_L \mathcal{C}_D, \\ y^{1-2\alpha} \frac{\partial w}{\partial \nu} = a(x)v & x \in D, y = 0. \end{cases} \quad (\text{A.7})$$

Corollary A.1. *Suppose $a, v \in C^\beta(D)$, $0 < \beta < 1$, are nonnegative functions. Then, for any $p > \frac{N}{N-2\alpha}$, there is a constant $C = C(p) > 0$, such that*

$$\|w(\cdot, 0)\|_{L^p(D)} \leq C \|a\|_{L^{\frac{N}{2\alpha}}(D)} \|v\|_{L^p(D)}. \quad (\text{A.8})$$

Proof. Let $f(x) = av$. For any $q > 1$, it follows from Proposition A.1 that

$$\|w(\cdot, 0)\|_{L^{\frac{Nq}{N-2\alpha q}}(D)} \leq C \|av\|_{L^q(D)} \leq C \|a\|_{L^{\frac{N}{2\alpha}}(D)} \|v\|_{L^{\frac{Nq}{N-2\alpha q}}(D)} \quad (\text{A.9})$$

We thus prove this corollary by letting $p = \frac{Nq}{N-2\alpha q}$. \square

Corollary A.2. *Let $w \in H_{0,L}^1(\mathcal{C}_D)$ be a solution of (A.7) with $a, v \geq 0$ and $a, v \in C^\beta(D)$. Then for any $\frac{N}{N-2\alpha} < p_2 < \frac{2N}{N-2\alpha}$, there is a constant $C = C(p_2) > 0$ such that*

$$\|w(\cdot, 0)\|_{L^{p_2}(D)} \leq C \|a\|_{L^r(D)} \|v\|_{L^{2_\alpha^*}(D)}, \quad (\text{A.10})$$

where $\frac{1}{p_2} = \frac{1}{r} + \frac{1}{2_\alpha^*} - \frac{2\alpha}{N}$.

Proof. Similar to the proof of Corollary A.1, we have

$$\|w(\cdot, 0)\|_{L^{p_2}(D)} \leq C \|av\|_{L^{\frac{p_2 N}{N+2\alpha p_2}}(D)} \leq C \|a\|_{L^r(D)} \|v\|_{L^{2_\alpha^*}(D)}, \quad (\text{A.11})$$

where r is determined by

$$\frac{1}{r} = \frac{N + 2\alpha p_2}{p_2 N} - \frac{1}{2_\alpha^*} = \frac{1}{p_2} + \frac{2\alpha}{N} - \frac{1}{2_\alpha^*}.$$

□

APPENDIX B. DECAY ESTIMATE

Consider the following problem:

$$\begin{cases} \operatorname{div}(y^{1-2\alpha} \nabla v) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\ y^{1-2\alpha} \frac{\partial v}{\partial y} = -|v(x, 0)|^{2_\alpha^*-2} v(x, 0), & \text{in } \mathbb{R}^N, \end{cases} \quad (\text{B.1})$$

In this section, we will obtain a decay estimate for solutions of (B.1).

Proposition B.1. *Suppose $v \in H_{0,L}^1(\mathbb{R}^{N+1})$ is a solution of (B.1), then there exists $C > 0$ such that*

$$|v(X)| \leq \frac{C}{(1 + |X|^2)^{\frac{N-2\alpha}{2}}} \quad (\text{B.2})$$

for $X \in \mathbb{R}_+^{N+1}$.

Before we prove Proposition B.1, we need the following lemma.

Lemma B.1. *For any $u \in C_0^\infty(\mathbb{R}^{N+1})$, it holds*

$$\int_{\mathbb{R}^{N+1}} |y|^{1-2\alpha} \frac{u^2}{|X|^2} dx dy \leq C \int_{\mathbb{R}^{N+1}} |y|^{1-2\alpha} |\nabla u|^2 dx dy.$$

Proof. This lemma may be known. Since the proof is short, we give the proof here.

Let

$$V(X) = \frac{|y|^{1-2\alpha}}{(N - 2\alpha)|X|^2} X.$$

Then

$$\operatorname{div} V = \frac{|y|^{1-2\alpha}}{|X|^2}.$$

Thus

$$\begin{aligned} \int_{\mathbb{R}^{N+1}} \frac{|y|^{1-2\alpha} u^2}{|X|^2} &= \int_{\mathbb{R}^{N+1}} u^2 \operatorname{div} V \\ &= - \int_{\mathbb{R}^{N+1}} 2u \nabla u \cdot V = -2 \int_{\mathbb{R}^{N+1}} u \nabla u \cdot \frac{|y|^{1-2\alpha} X}{(N-2\alpha)|X|^2} \\ &\leq \frac{2}{N-2\alpha} \left(\int_{\mathbb{R}^{N+1}} |y|^{1-2\alpha} |\nabla u|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^{N+1}} |y|^{1-2\alpha} \frac{u^2}{|X|^2} \right)^{\frac{1}{2}}. \end{aligned}$$

□

Proof of Proposition B.1. To prove (B.2), we use the following Kelvin transformation

$$\tilde{v}(X) = |X|^{-N+2\alpha} v\left(\frac{X}{|X|^2}\right)$$

of v . If v is a solution of (B.1), then \tilde{v} satisfies

$$\begin{cases} \operatorname{div}(y^{1-2\alpha} \nabla \tilde{v}) = 0, & \text{in } \mathbb{R}_+^{N+1} \setminus \{0\}, \\ y^{1-2\alpha} \frac{\partial \tilde{v}}{\partial y} = -|\tilde{v}(x, 0)|^{2_\alpha^*-2} \tilde{v}(x, 0), & \text{in } \mathbb{R}^N \setminus \{0\}, \end{cases} \quad (\text{B.3})$$

Moreover, we have

$$\int_{\mathbb{R}^N} |\tilde{v}(x, 0)|^{2_\alpha^*} dx \leq C. \quad (\text{B.4})$$

On the other hand, it follows from Lemma B.1 that

$$\int_{\mathbb{R}_+^{N+1}} y^{1-2\alpha} |\nabla \tilde{v}|^2 dx dy \leq C. \quad (\text{B.5})$$

From (B.4) and (B.5), it is standard to prove that \tilde{v} is a solution of (B.1). Harnack inequality gives

$$|\tilde{v}| \leq C, \quad \text{in } \mathcal{B}_1(0) \cap \mathbb{R}^{N+1}.$$

Hence, (B.2) follows.

□

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